UG-07-01

Scaling cosmologies, geodesic motion and pseudo-susy

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Abstract

One-parameter solutions in supergravity carried by scalars and a metric trace out curves on the scalar manifold. In ungauged supergravity these curves describe a geodesic motion. It is known that a geodesic motion sometimes occurs in the presence of a scalar potential and for time-dependent solutions this can happen for scaling cosmologies. This note contains a further study of such solutions in the context of pseudo-supersymmetry for multi-field systems whose first-order equations we derive using a Bogomol'nyi-like method. In particular we show that scaling solutions that are pseudo-BPS must describe geodesic curves. Furthermore, we clarify how to solve the geodesic equations of motion when the scalar manifold is a maximally non-compact coset such as occurs in maximal supergravity. This relies upon a parametrization of the coset in the Borel gauge. We then illustrate this with the cosmological solutions of higher-dimensional gravity compactified on a n-torus.

Contents

1 Preliminaries

We consider scalar fields Φ^i that parametrize a Riemannian manifold with metric G_{ij} coupled to gravity through the standard action

$$S = \int \sqrt{-g}\Big\{\mathcal{R} - \tfrac{1}{2}G_{ij}g^{\mu\nu}\partial_\mu\Phi^i\partial_\nu\Phi^j - V(\Phi)\Big\}\,. \tag{1}$$

We restrict to solutions with the following D-dimensional space-time metric

$$\mathrm{d}s^2_D = g(y)^2\mathrm{d}s^2_{D-1} + \epsilon f(y)^2\mathrm{d}y^2\,, \quad \mathrm{d}s^2_{D-1} = (\eta_\epsilon)_{ab}\mathrm{d}x^a\mathrm{d}x^b\,, \tag{2}$$

where $\epsilon = \pm 1$ and $\eta_\epsilon = \mathrm{diag}(-\epsilon, 1, \ldots, 1)$. The case $\epsilon = -1$ describes a flat FLRW-space-time and $\epsilon = +1$ a Minkowski-sliced domain wall (DW) space-time. The scalar fields that source these space-times can only depend on the y-coordinate $\Phi^i = \Phi^i(y)$. The function f corresponds to the gauge freedom of reparameterizing the y-coordinate.

Of particular interest in this note are scaling comologies, which have received a great deal of attention in the dark-energy literature, see [1] for a review and references. One definition (amongst many) of scaling cosmologies is that they are solutions for which all terms in the Friedmann equation have the same time dependence. For pure scalar cosmologies this implies that

$$H^2 \sim V \sim T \sim \tau^{-2}\,, \tag{3}$$

where τ denotes cosmic time, H the Hubble parameter and T is the kinetic energy $T = \frac{1}{2}G_{ij}\dot\Phi^i\dot\Phi^j$. These relations imply that the scale factor is power-law $a(\tau) \sim \tau^p$. In the case of curved FLRW-universes we also demand that $H \sim k/a^2$, which is only possible for $p = 1$. Interestingly, scaling solutions correspond to the FLRW-geometries that possess a time-like conformal vectorfield ξ coming from the transformation

$$\tau \to \mathrm{e}^{\lambda}\tau\,, \qquad x^a \to \mathrm{e}^{(1-p)\lambda}x^a\,, \tag{4}$$

where x^a are the space-like cartesian coordinates[1]. In the forthcoming we reserve the indices $a, b, \ldots$ to denote space-like coordinates when we consider cosmological space-times. Apart from the intriguing cosmological properties of scaling solutions they are also interesting for understanding the dynamics of a general cosmological solution since scaling solutions are often critical points of an autonomous system of differential equations and therefore correspond to attractors, repellors or saddle points [2]. Scaling cosmologies often appear in supergravity theories (see for instance [3,4]) but, remarkably, they also appear by spatially averaging inhomogeneous cosmologies in classical general relativity [5].

We will use two coordinate frames to describe scaling comologies

$$\tau - \text{frame}: \qquad \mathrm{d}s^2 = -\mathrm{d}\tau^2 + \tau^{2p}\,\mathrm{d}s^2_{D-1}\,, \tag{5}$$

$$t - \text{frame}: \qquad \mathrm{d}s^2 = -\mathrm{e}^{2t}\,\mathrm{d}t^2 + \mathrm{e}^{2pt}\mathrm{d}s^2_{D-1}\,. \tag{6}$$

The first is the usual FLRW-coordinate system and the second can be obtained by the substitution $t = \ln\tau$.

2 (Pseudo-) supersymmetry

If the scalar potential $V(\Phi)$ can be written in terms of another function $W(\Phi)$ as follows

$$V = \epsilon\left\{\tfrac{1}{2}G^{ij}\partial_i W\partial_j W - \tfrac{D-1}{4(D-2)}W^2\right\}, \tag{7}$$

then the action can be written as "a sum of squares" plus a boundary term when reduced to one dimension:

$$\begin{aligned} S =& \epsilon\int \mathrm{d}y\, fg^{D-1}\Big\{\tfrac{(D-1)}{4(D-2)}\Big[W - 2(D-2)\frac{\dot g}{fg}\Big]^2 - \tfrac{1}{2}||\frac{\dot\Phi^i}{f} + G^{ij}\partial_j W||^2\Big\} \\ &+ \epsilon\int \mathrm{d}\Big\{g^{D-1}W - 2(D-1)\dot g g^{D-2}f^{-1}\Big\}\,, \end{aligned} \tag{8}$$

where a dot denotes a derivative w.r.t. y. The term $||\dot\Phi^i/f + G^{ij}\partial_j W||^2$ is a shorthand notation and the square involves a contraction with the field metric G_{ij}. It is clear that the action is stationary under variations if the terms within brackets are zero[2], leading to the following *first-order* equations of motion

$$\boxed{W = 2(D-2)\frac{\dot g}{fg}\,, \qquad \frac{\dot\Phi^i}{f} + G^{ij}\partial_j W = 0\,.} \tag{9}$$

[1] For curved FLRW-space-times the space-like coordinates are invariant.

[2] For completeness we should have added the Gibbons-Hawking term [6] in the action which deletes that part of the above boundary term that contains $\dot g$.

For $\epsilon = +1$ these equations are the standard Bogomol'nyi-Prasad-Sommerfield (BPS) equations for domain walls that arise from demanding the susy-variation of the fermions to vanish, which guarantees that the DW preserves a fraction of the total supersymmetry of the theory. The function W is then the superpotential that appears in the susy-variation rules and equation (7) with $\epsilon = +1$ is natural for supergravity theories. It is clear that for every W that obeys (7) we can find a corresponding DW-solution, and if W is not related to the susy-variations we call the solutions fake supersymmetric [7].

For $\epsilon = -1$ these equations are the generalization to arbitrary space-time dimension D and field metric G_{ij} of the framework found in references [8–11]. So here we generalized and derived in a different way (some of) the results of [8–11] by showing that analogously to DW's we can write the Lagrangian as a sum of squares. We refer to these first-order equations as pseudo-BPS equations and W is named the pseudo-superpotential because of the immediate analogy with BPS domain walls in supergravity [10, 11]. For the case of cosmologies there is no natural choice for W as cosmologies cannot be found by demanding vanishing susy-variations of the fermions[3].

In [11] it is proven that for all single-scalar cosmologies (and domain walls) a pseudo-superpotential W exists such that the cosmology is pseudo-BPS and that one can give a fermionic interpretation of the pseudo-BPS flow in terms of so-called pseudo-Killing spinors. This does not necessarily carry over to multi-scalar solutions as was shown in [14]. Nonetheless, a multi-field solution can locally be seen as a single-field solution [15] because locally we can redefine the scalar coordinates such that the curve $\Phi(y)$ is aligned with a scalar axis and all other scalars are constant on this solution. A necessary condition for the single-field pseudo-BPS flow to carry over (locally) to the multi-field system is that the truncation down to a single scalar is consistent (this means that apart from the solution one can put the other scalars always to zero) [14].

3 Multi-field scaling cosmologies

Let us turn to scaling solutions in the framework of pseudo-supersymmetry and see how geodesic motion arises. First we consider the rather trivial case with vanishing scalar potential V and then in section 3.2 we add a scalar potential V. Pseudo-supersymmetry is only discussed in the case of non-vanishing V.

3.1 Pure kinetic solutions

If there is no scalar potential the solutions trace out geodesics since after a change of coordinates $y \to \tilde{y}(y)$ via $\mathrm{d}\tilde{y} = fg^{1-D}\mathrm{d}y$, the scalar field action becomes $\int G_{ij}\Phi'^i\Phi'^j\mathrm{d}\tilde{y}$, where a prime means a derivative w.r.t. $\tilde{y}$. This new action describes geodesic curves with affine parameter $\tilde{y}$. The affine velocity is constant by definition and positive since the metric is positive definite

$$G_{ij}\Phi'^i\Phi'^j = ||v||^2\,. \tag{10}$$

[3]Star supergravity is an exception [12] and that seems related to pseudo-supersymmetry [13].

The Einstein equation is

$$\mathcal{R}_{yy} = \tfrac{1}{2} G_{ij} \dot{\Phi}^i \dot{\Phi}^j = \frac{||v||^2}{2} g^{2-2D} f^2 \,, \qquad \mathcal{R}_{ab} = 0 \,. \tag{11}$$

In the gauge $f = 1$ the solution is given by $g = \mathrm{e}^{C_2} (y + C_1)^{\overline{\frac{1}{D-1}}}$, with C_1 and C_2 arbitrary integration constants, but with a shift of y we can always put $C_1 = 0$ and C_2 can always be put to zero by re-scaling the space-like coordinates. In the case of a four-dimensional cosmology the geometry is a power-law FLRW-solution with $p = 1/3$.

3.2 Potential-kinetic scaling solutions

In a recent paper of Tolley and Wesley an interesting interpretation was given to scaling solutions [16], which we repeat here. The finite transformation (4) leaves the equations of motion invariant if the action S scales with a constant factor, which is exactly what happens for scaling solutions since all terms in the Lagrangian scale like τ^{-2}. Under (4) the metric scales like $\mathrm{e}^{2\lambda} g_{\mu\nu}$ and in order for the action to scale as a whole we must have

$$V \to \mathrm{e}^{-2\lambda} V \,, \qquad T = \tfrac{1}{2} g^{\tau\tau} G_{ij} \dot{\Phi}^i \dot{\Phi}^j \to \mathrm{e}^{-2\lambda} T \,. \tag{12}$$

Equations (12) imply that $G_{ij} \dot{\Phi}^i \dot{\Phi}^j$ remains invariant from which one deduces that $\frac{\mathrm{d}\Phi^i}{\mathrm{d}\lambda} = \xi^i$ must be a Killing vector. The curve that describes a scaling solution follows an isometry of the scalar manifold. It depends on the parametrization whether the tangent vector $\dot{\Phi}$ itself is Killing. This happens for the parametrization in terms of $t = \ln \tau$ since

$$\xi^i = \frac{\mathrm{d}\Phi^i}{\mathrm{d}\lambda} = \lim_{\lambda \to 0} \frac{\Phi^i(e^\lambda \tau) - \Phi^i(\tau)}{\lambda} = \frac{\mathrm{d}\Phi^i}{\mathrm{d}\ln\tau} \,. \tag{13}$$

Thus a scaling solution is associated with an invariance of the equations of motion for a rescaling of cosmic time and is therefore associated with a conformal Killing vector on space-time and a Killing vector on the scalar manifold.

Pseudo-supersymmetry comes into play when we check the geodesic equation of motion

$$\nabla_{\dot{\Phi}} \dot{\Phi}_i = \dot{\Phi}^j \nabla_j \dot{\Phi}_i = \dot{\Phi}^j \Big\{ \nabla_{(j} \dot{\Phi}_{i)} + \nabla_{[j} \dot{\Phi}_{i]} \Big\} \,, \tag{14}$$

where we denote $\dot{\Phi}_i = G_{ik} \dot{\Phi}^k$. Now we have that the symmetric part is zero if we parametrize the curve with $t = \ln \tau$ since scaling makes $\dot{\Phi}$ a Killing vector. We also have that $\nabla_{[j} \dot{\Phi}_{i]} = 0$ since the pseudo-BPS condition makes $\dot{\Phi}$ a curl-free flow $\dot{\Phi}_i = -f \partial_i W$. To check that the curl is indeed zero (when $f \neq 1$) one has to notice that in the parametrization of the curve in terms of $t = \ln \tau$ the gauge is such that $\dot{g}/g$ is constant and that $f \sim W^{-1}$. Since the curl is also zero we notice that the curve is a geodesic with $\ln \tau$ as affine parametrization[4]

$$\nabla_{\dot{\Phi}} \dot{\Phi}^i = 0 = \ddot{\Phi}^i + \Gamma^i_{jk} \dot{\Phi}^j \dot{\Phi}^k \,. \tag{15}$$

[4] One could wonder whether the results works in two ways. Imagine that a scaling solution is a geodesic. This then implies that $\nabla_{[j} \dot{\Phi}_{i]} = 0$ and therefore the flow is locally a gradient flow $\dot{\Phi}_i = \partial_i \ln W \sim f \partial_i W$.

The link between scaling and geodesics was discovered by Karthauser and Saffin in [17], but no conditions on the Lagrangian were given in [17] such that the relation scaling-geodesic holds. An example of a scaling solution that is not a geodesic was given by Sonner and Townsend in [18].

A more intuitive understanding of the origin of the geodesic motion for some scaling cosmologies comes from the on-shell substitution $V = (3p-1)\,T$ in the Lagrangian to get a new Lagrangian describing seemingly massless fields. Although this is rarely a consistent procedure we believe that this is nonetheless related to the existence of geodesic scaling solutions.

Single field

For single-field models the potential must be exponential $V = \Lambda \mathrm{e}^{\alpha\phi}$ in order to have scaling solutions. The simplest pseudo-superpotential belonging to an exponential potential is itself exponential

$$W = \pm\sqrt{\tfrac{8\Lambda}{3-\alpha^2}}\;\mathrm{e}^{\frac{\alpha\phi}{2}}\,. \tag{16}$$

If we choose the plus sign the solution to the pseudo-BPS equation is

$$\phi(\tau) = -\tfrac{2}{\alpha}\ln\tau + \tfrac{1}{\alpha}\ln[\tfrac{6-2\alpha^2}{\alpha^4\Lambda}]\,, \qquad g(\tau) \sim \tau^{\frac{1}{\alpha^2}}\,. \tag{17}$$

The minus sign corresponds to the time reversed solution.

Multiple fields

For a general multi-field model a scaling solution with power-law scale factor τ^p obeys $V = (3p-1)T$ from which we derive the **on-shell** relation

$$G^{ij}\partial_i W\partial_j W = \frac{W^2}{4p} \quad\Rightarrow\quad W = \pm\sqrt{\frac{8\,p\,V}{3p-1}}\,. \tag{18}$$

In general the above expression for the superpotential $W \sim \sqrt{V}$ does not hold off-shell, unless the potential is a function of a specific kind:

$$\frac{1}{p} = \frac{G^{ij}\partial_i V\partial_j V}{V^2}\,. \tag{19}$$

Scalar potentials that obey (19) with the extra condition that $p \gtrless \frac{1}{3} \leftrightarrow V \gtrless 0$ allow for multi-field scaling solutions. For a given scalar potential that obeys (19) there probably exist many pseudo-superpotentials W compatible with V but if we make the specific choice $W = \sqrt{8\,p\,V/(3p-1)}$ then all pseudo-BPS solutions must be scaling and hence geodesic. As a consistency check we substitute the first-order pseudo-BPS equations into the right-hand-side of the following second-order equations of motion

$$\ddot{\Phi}^i + \Gamma^i_{jk}\dot{\Phi}^k\dot{\Phi}^j = -f^2 G^{ij}\partial_j V - \Big[3(\ln\dot{g}) - (\ln\dot{f})\Big]\dot{\Phi}^i\,, \tag{20}$$

and choose a gauge for which

$$\frac{\dot{f}}{f^2} = \frac{1}{4p} W \,, \tag{21}$$

then we indeed find an affine geodesic motion since the right hand side of (20) vanishes.

For some systems one first needs to perform a truncation in order to find the above relation (19). A good example is the multi-field potential appearing in Assisted Inflation [19]

$$V(\Phi^1, \ldots, \Phi^n) = \sum_i^n \Lambda_i \, \mathrm{e}^{\alpha_i \Phi^i} \,, \qquad G_{ij} = \delta_{ij} \,. \tag{22}$$

The scaling solution of this system was proven to be the same as the single-exponential scaling [20]. The reason is that one can perform an orthogonal transformation in field space such that the form of the kinetic term is preserved but the scalar potential is given by

$$V = \mathrm{e}^{\alpha\varphi} \, U(\Phi^1, \ldots, \Phi^{n-1}) \,, \quad \frac{1}{\alpha^2} = \sum_i \frac{1}{\alpha_i^2} \,. \tag{23}$$

The scaling solution is such that $\Phi_1, \ldots, \Phi_{n-1}$ are frozen in a stationary point of U and therefore the system is truncated to a single-field system that obeys (19). The same was proven for Generalized Assisted Inflation [21] in reference [22]. The scaling solution in the original field coordinates reads $\Phi^i = A^i \ln \tau + B^i$, which is clearly a straight line and thus a geodesic.

The scaling solutions of [14, 18] were constructed for an axion-dilaton system with an exponential potential for the dilaton

$$S = \int \sqrt{-g} \Big\{ \mathcal{R} - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}\mathrm{e}^{\mu\phi}(\partial\chi)^2 - \Lambda \mathrm{e}^{\alpha\phi} \Big\} \,. \tag{24}$$

Clearly this two-field system obeys (19) and (one of) the pseudo-superpotential(s) is given by (16). The pseudo-BPS scaling solution therefore has constant axion and is effectively described by the dilaton in an exponential potential. Note that this solution indeed describes a geodesic on $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$ with $\ln\tau$ as affine parameter. All examples of scaling solutions in the literature seem to occur for exponential potentials, however by performing a $\mathrm{SL}(2,\mathbb{R})$-transformation on the Lagrangian (24) the kinetic term is unchanged and the potential becomes a more complicated function of the axion and the dilaton. The same scaling solution then trivially still exists (and (19) still holds) but the axion is not constant in the new frame and instead the solution follows a more complicated geodesic on $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$.

However another scaling solution is given in [18] that is not geodesic and with varying axion in the frame of the above action (24). This is an illustration of the above, since the solution is not geodesic we know that there does not exists any other pseudo-superpotential for which the varying axion solution is pseudo-BPS, consistent with what is shown in [14] for that particular solution.

4 Geodesic curves and the Borel gauge

For the last example of the previous section the pseudo-BPS scaling solutions described geodesics on the symmetric space $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$. In this section we consider a general class of symmetric spaces of which $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$ is an example and they are known as maximally non-compact cosets U/K. It seems that for this class of spaces the geodesic equations of motion can be solved easily. The symmetry of the geodesic equations is the symmetry of the scalar coset U/K. In the case of maximal supergravity the symmetry U is a U-duality and is a maximal non-compact real slice of a complex semisimple group. The isotropy group K is the maximal compact subgroup of U.

4.1 A solution-generating technique

In the *Borel gauge* the scalar fields are divided into r dilatons ϕ^I and $(n-r)$ axions χ^α, with r the rank of U and n the dimension of U/K (see for instance [23]). The dilatons are related to the generators H_I of the Cartan sub-algebra (CSA) and the axions to the positive root generators E_α through the following expression for the coset representative L in the Borel gauge

$$L = \Pi_\alpha \exp[\chi^\alpha E_\alpha]\Pi_I \exp[-\tfrac{1}{2}\phi^I H_I]\,. \tag{25}$$

In this language the geodesic equation is

$$\ddot{\phi}^I + \Gamma^I_{JK}\dot{\phi}^J\dot{\phi}^K + \Gamma^I_{\alpha J}\dot{\chi}^\alpha\dot{\phi}^J + \Gamma^I_{\alpha\beta}\dot{\chi}^\alpha\dot{\chi}^\beta = 0\,, \tag{26}$$

$$\ddot{\chi}^\alpha + \Gamma^\alpha_{JK}\dot{\phi}^J\dot{\phi}^K + \Gamma^\alpha_{\beta J}\dot{\chi}^\beta\dot{\phi}^J + \Gamma^\alpha_{\beta\gamma}\dot{\chi}^\beta\dot{\chi}^\gamma = 0\,. \tag{27}$$

Since $\Gamma^I_{JK} = 0$ and $\Gamma^\alpha_{JK} = 0$ at points for which $\chi^\alpha = 0$ a trivial solution is given by

$$\phi^I = v^I\, y\,, \qquad \chi^\alpha = 0\,. \tag{28}$$

How many other solutions are there? A first thing we notice is that every global U-transformation $\Phi \to \tilde{\Phi}$ brings us from one solution to another solution. Since U generically mixes dilatons and axions we can construct solutions with non-trivial axions in this way. We now prove that in this way *all* geodesics are obtained and this depends on the fact that U is maximally non-compact with K the maximal compact subgroup of U.

Consider an arbitrary geodesic curve $\Phi(t)$ on U/K. The point $\Phi(0)$ can be mapped to the origin $L = \mathbb{1}$ using a U-transformation, since we can identify $\Phi(0)$ with an element of U and then we multiply the geodesic curve $\Phi(t)$ with $\Phi(0)^{-1}$, generating a new geodesic curve $\Phi_2(t) = \Phi(0)^{-1}\Phi(t)$ that goes through the origin. The origin is invariant under K-rotations but the tangent space at the origin transforms under the adjoint of K. One can prove that there always exists an element $k \in K$, such that $\mathrm{Adj}_k\dot{\Phi}_2(0) \in \mathrm{CSA}$ [24]. Therefore $\dot{\chi}_2^\alpha = 0$ and this solution must be a straight line. So we started out with a general curve $\Phi(t)$ and proved that the curve $\Phi_3(t) = k\Phi(0)^{-1}\Phi(t)$ is a straight line.

4.2 An illustration from dimensional reduction

The metric Ansatz for the dimensional reduction of $(4+n)$-dimensional Einstein-gravity on the n-torus ($\mathbb{T}^n$) is

$$\mathrm{d}s^2_{4+n} = e^{2\alpha\varphi}\mathrm{d}s^2_4 + e^{2\beta\varphi}\mathcal{M}_{ab}\mathrm{d}z^a \otimes \mathrm{d}z^b\,, \tag{29}$$

where

$$\alpha^2 = \frac{n}{4(n+2)}\,, \qquad \beta = -\frac{2\alpha}{n}\,. \tag{30}$$

The matrix $\mathcal{M}$ is a positive-definite symmetric $n\times n$ matrix with unit determinant, which depends on the 4-dimensional coordinates, describing the moduli of $\mathbb{T}^n$. The modulus φ controls the overall volume and is named the breathing mode or radion field. Notice that we already truncated the Kaluza–Klein vectors in the Ansatz. The reduction of the Einstein–Hilbert term gives

$$\mathcal{L} = \sqrt{-g}\{\mathcal{R} - \tfrac{1}{2}(\partial\varphi)^2 + \tfrac{1}{4}\mathrm{Tr}\partial\mathcal{M}\partial\mathcal{M}^{-1}\}\,. \tag{31}$$

The scalars parametrize $\mathbb{R}\times \mathrm{SL}(n,\mathbb{R})/\,\mathrm{SO}(n)$ where φ belongs to the decoupled $\mathbb{R}$-part and $\mathcal{M}$ is the $\mathrm{SL}(n,\mathbb{R})/\,\mathrm{SO}(n)$ part.

If we take the four-dimensional part of space-time to be a flat FLRW-space then that part of the metric will be power-law with $p=1/3$ and the scalars follow a geodesic with $\ln\tau$ as an affine parameter. According to the solution-generating technique, the Ansatz for the scalars is

$$\varphi = v_0\ln\tau + c_0\,,\quad \mathcal{M} = \Omega D\Omega^T\,,\quad D = \mathrm{diag}(\mathrm{e}^{-\vec{\beta}_a\cdot\vec{\phi}})\,, \tag{32}$$

with $\vec{\phi} = \vec{v}\ln\tau$ and $\vec{\beta}$ the weights of $\mathrm{SL}(n,\mathbb{R})$ in the fundamental representation (see appendix B for some explanations on the $\mathrm{SL}(n,\mathbb{R})/\,\mathrm{SO}(n)$-coset in this representation). The diagonal matrix D represents the straight-line solution and Ω is an arbitrary $\mathrm{SL}(n,\mathbb{R})$-matrix in the fundamental representation. Therefore $\mathcal{M}=\Omega D\Omega^T$ is the most general coset matrix describing a geodesic curve.

The Friedmann equation implies that the affine velocity is restricted to be

$$v_0^2 + ||v||^2 = \tfrac{4}{3}, \tag{33}$$

which is the only constraint coming from the 4-dimensional Einstein equation. If we substitute this solution in (29) and define new coordinates $\vec{y} = \vec{z}\,\Omega$ we find

$$\mathrm{d}s^2_{4+n} = -\tau^{2\alpha\,v_0}\mathrm{d}\tau^2 + \tau^{\frac{2}{3}+2\alpha\,v_0}\mathrm{d}\vec{x}^2_3 + \sum_{a=1}^{n}\tau^{-\vec{\beta}_a\cdot\vec{v}+2\beta\,v_0}\mathrm{d}y_a^2\,. \tag{34}$$

This is similar to what is called a Kasner solution in general relativity (see for instance [25]). Kasner solutions are a general class of time-dependent geometries that look like

$$\mathrm{d}s^2 = -\tau^{2p_0}\mathrm{d}\tau^2 + \sum_a \tau^{2p_a}\mathrm{d}x_a^2\,. \tag{35}$$

Kasner solutions solve the Einstein equations in vacuum if the following two conditions are satisfied

$$p_0 + 1 = \sum_a p_a\,, \quad (p_0+1)^2 = \sum_a p_a^2\,. \tag{36}$$

For the metric (34) these conditions are satisfied if the lower-dimensional Friedmann equation is satisfied. For this calculation one needs the properties of the weight-vectors $\vec{\beta}_a$ (given in appendix B) and the relation between α and β (30). We therefore conclude that the general spatially flat FLRW-solution lifts up to the most general Kasner solution with SO(3)-symmetry in $4+n$ dimensions.

5 Discussion

In this note we have studied multi-field scaling solutions using a first-order formalism for scalar cosmologies *a.k.a.* pseudo-supersymmetry. We derived these first-order equations via a Bogomol'nyi-like method that was known to work for domain wall solutions as was first shown in [26, 27][5] and we showed that it trivially extends to cosmological solutions. This first-order formalism allows a better understanding of the geodesic motion that comes with a specific class of scaling solutions. One of the main results of this note is a proof that shows that all pseudo-BPS cosmologies that are scaling solutions must be geodesic. This complements to the discussion in [14] where the first example of a non-geodesic scaling cosmology was shown to be non-pseudo-BPS. Moreover we gave constraints on multi-field Lagrangians for which the pseudo-BPS cosmologies are geodesic scaling solutions.

Having illustrated the importance of geodesic motion in scalar cosmology, we tackled the problem of solving the geodesic equations in the second part of this note. We showed that the most general geodesic curve can be written down for maximally non-compact coset spaces U/K. These coset spaces appear in all maximal and some less-extended supergravities [29]. We used a solution-generating technique based on the symmetries of the coset. We were able to prove that the most general solution is given by a U-transformation on the "straight line", $(\phi^I(t) = v^I t, \chi^\alpha = 0)$ in the Borel gauge. We illustrated this technique for the coset $\mathrm{SL}(n,\mathbb{R})/\,\mathrm{SO}(n)$. Since $\mathrm{SL}(n,\mathbb{R})/\,\mathrm{SO}(n)$ is also the moduli space of the n-torus we applied it to find the cosmological solutions of higher-dimensional gravity compactified on a n-torus. This exercise nicely illustrates why the straight line is the generating solution since, from a higher-dimensional point of view, all solutions that correspond to the non-straight line geodesics can be seen as coordinate transformations of the solutions associated with the straight line. The oxidation of the straight line solutions corresponds to the most general SO(3)-invariant Kasner solution of $(4+n)$-dimensional vacuum GR.

The same technique was used in [3] to find all geodesic scaling cosmologies of the CSO-gaugings in maximal supergravity.

The solution-generating technique presented here should be considered complementary to the "compensator method" developed by Fré et al in [30]. There the straight line

[5] See also [28].

also serves as a generating solution but instead of rigid U-transformations one uses local K transformations that preserve the solvable gauge to generate new non-trivial solutions. This technique is a nice illustration of the integrability of the second–order geodesic equations of motion [31].

Acknowledgments

We are grateful to Dennis Westra for useful discussions and comments on the manuscript and to Jan Rosseel for many useful discussions. This work is supported in part by the European Communitys Human Potential Programme under contract MRTN-CT-2004-005104 in which the authors are associated to Utrecht University. The work of AP and TVR is part of the research programme of the Stichting voor Fundamenteel Onderzoek der Materie (FOM).

A Curvatures

For the metric Ansatz (2) the Ricci tensor is given by

$$\mathcal{R}_{ab} = -\epsilon(\eta_\epsilon)_{ab}\Big\{\frac{\mathrm{d}}{\mathrm{d}y}[\frac{g\dot{g}}{f^2}] + \frac{g\dot{g}\dot{f}}{f^3} + (D-3)\frac{\dot{g}^2}{f^2}\Big\}\,, \qquad \mathcal{R}_{yy} = (D-1)\Big\{-(\frac{\ddot{g}}{g}) + \frac{\dot{g}\dot{f}}{gf}\Big\}\,. \tag{37}$$

B The coset $\mathrm{SL}(N,\mathrm{I\!R})/\,\mathrm{SO}(N)$

Consider a general coset U/K. It is not difficult to construct a coset representative using the Lie algebras $\mathfrak{U}$ and $\mathfrak{K}$ of U and K respectively. Since K is a subgroup of U we have the decomposition $\mathfrak{U} = \mathfrak{K} \oplus \mathfrak{F}$, with $\mathfrak{F}$ the complement of $\mathfrak{K}$ in $\mathfrak{U}$. For a given representation of the algebra $\mathfrak{U}$ we define a coset representative via $L(y) = \exp(y^i\mathbf{f_i})$ where the $\mathbf{f_i}$ form a basis of $\mathfrak{F}$ in some representation of $\mathfrak{U}$.

To derive the metric we define a Lie algebra valued one-form from the coset representative $L(y)$ via

$$L^{-1}\mathrm{d}L \equiv E + \Omega\,, \tag{38}$$

where E takes values in $\mathfrak{F}$ and Ω in $\mathfrak{K}$. We notice that $L^{-1}\mathrm{d}L$ is invariant under left multiplication with a y-independent element $g \in U$. Multiplying L from the right with local elements $k \in K$ results in

$$E \to k^{-1}\,E\,k\,, \qquad \Omega \to k^{-1}\,\Omega\,k + k^{-1}\mathrm{d}k\,. \tag{39}$$

In supergravity the parameters y^i are scalar fields that depend on the space-time coordinates $y^i = \phi^i(x)$. The one-form $L^{-1}\mathrm{d}L$ can be written out in terms of coset-coordinate one-forms $\mathrm{d}\phi^i$ which themselves can be pulled back to space-time coordinate one-forms $\mathrm{d}\phi^i = \partial_\mu\phi^i\mathrm{d}x^\mu$. Now we can write

$$L^{-1}\mathrm{d}L = E_\mu\mathrm{d}x^\mu + \Omega_\mu\mathrm{d}x^\mu\,. \tag{40}$$

Under the ϕ-dependent K-transformations $k(\phi(x))$ we have that $\Omega_\mu \to k^{-1}\Omega_\mu k + k^{-1}\partial_\mu k$ and $E_\mu \to k^{-1}E_\mu k$. It is clear that E_μ is covariant under local K-transformations and Ω_μ transforms like a connection. Using this connection Ω_μ we can make the following K-covariant derivative on L and L^{-1}

$$D_\mu L = \partial_\mu L - L\Omega_\mu\,, \qquad D_\mu L^{-1} = \partial_\mu L^{-1} + \Omega_\mu L^{-1}\,. \tag{41}$$

To find a kinetic term for the scalars we notice that the object

$$\mathrm{Tr}[D_\mu L D^\mu L^{-1}] = -\mathrm{Tr}[E_\mu E^\mu]\,, \tag{42}$$

has all the right properties as it contains single derivatives on the scalars, it is a space-time scalar, it is invariant under rigid U transformations and under local K-transformations. Thus,

$$e^{-1}\mathcal{L}_{\text{scalar}} = -\mathrm{Tr}[E_\mu E^\mu] \equiv -\tfrac{1}{2}g(\phi)_{ij}\partial_\mu\phi^i\partial^\mu\phi^j\,. \tag{43}$$

If $\mathrm{SO}(N)$ is the maximal compact subgroup of U and we work in the fundamental representation, then the Lie algebra of $\mathrm{SO}(N)$ is the vector space of antisymmetric matrices,

$$E = \frac{L^{-1}\mathrm{d}L + (L^{-1}\mathrm{d}L)^T}{2}\,, \qquad \Omega = \frac{L^{-1}\mathrm{d}L - (L^{-1}\mathrm{d}L)^T}{2}\,, \tag{44}$$

and a calculation shows that

$$e^{-1}\mathcal{L}_{\text{scalar}} = -\mathrm{Tr}[E^2] = +\tfrac{1}{4}\mathrm{Tr}[\partial\mathcal{M}\partial\mathcal{M}^{-1}]\,, \tag{45}$$

where $\mathcal{M}$ is the $\mathrm{SO}(N)$-invariant matrix $\mathcal{M} = LL^T$.

No we specify to $U = \mathrm{SL}(N, \mathrm{I\!R})$. In general $\mathrm{SL}(N, \mathrm{I\!R})$ has rank $N-1$ and its maximal compact subgroup is $\mathrm{SO}(N)$. There will therefore be $N-1$ dilaton fields ϕ^I and $N(N-1)/2$ axion fields χ^α. The Cartan generators are given in terms of the weights $\vec{\beta}$ of $\mathrm{SL}(N, \mathrm{I\!R})$ in the fundamental representation

$$(\vec{H})_{ij} = (\vec{\beta}_i)\delta_{ij}\,. \tag{46}$$

The weights can be taken to obey the following algebra

$$\sum_i \beta_{iI} = 0\,, \quad \sum_i \beta_{iI}\beta_{iJ} = 2\delta_{IJ}\,, \quad \vec{\beta}_i \cdot \vec{\beta}_j = 2\delta_{ij} - \frac{2}{N}\,. \tag{47}$$

The first of these identities holds in all bases since it follows from the tracelessness of the SL generators. The second and third identity can be seen as convenient normalizations of the generators. The positive step operators E_{ij} are all upper triangular and a handy basis is that they have only one non-zero entry $[E_{ij}]_{ij} = 1$. The negative step operators are the transpose of the positive. The $\mathrm{SO}(N)$ algebra is spanned by the following combinations

$$\frac{1}{\sqrt{2}}(E_\beta - E_{-\beta})\,. \tag{48}$$

The action will generically look complicated but when all axions are set to zero L is diagonal $L = \mathrm{diag}[\exp(-\tfrac{1}{2}\vec{\beta}_i \cdot \vec{\phi})]$ and the action becomes

$$+\tfrac{1}{4}\mathrm{Tr}\partial\mathcal{M}\partial\mathcal{M}^{-1} = -\tfrac{1}{4}(\sum_i \beta_{iJ}\beta_{iI})\partial\phi^I\partial\phi^J = -\tfrac{1}{2}\delta_{IJ}\partial\phi^I\partial\phi^J\,. \tag{49}$$

This action describes $N-1$ dilatons that parametrize the flat scalar manifold $\mathrm{I\!R}^{N-1}$.

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